

GENERALIZATIONS OF BORSUK-ULAM THEOREM

Dedicated to Professor Keizo Asano on his 60th birthday

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(Received April 10, 1970)

Introduction

Conner-Floyd proved in their book [1] the following theorem which is a generalization of the classical Borsuk-Ulam theorem: Let $f: S^n \rightarrow M$ be a continuous map of the n -sphere to a differentiable manifold of dimension m , and T be a fixed point free differentiable involution on S^n . Assume that $m \leq n$ and $f_*: H_n(S^n; \mathbb{Z}_2) \rightarrow H_n(M; \mathbb{Z}_2)$ is trivial. Then the covering dimension of $A(f) = \{y \in S^n \mid f(y) = f(Ty)\}$ is at least $n - m$.

In response to the questions asked in [1, p. 89], Munkholm [4] showed that in the above theorem all differentiability hypotheses can be eliminated if M is assumed to be compact. Furthermore he showed in [5] that S^n can be replaced by a closed manifold which is a mod 2 homology n -sphere if M is the Euclidean space. In the present paper, we shall show the following theorem which is more general.

Main Theorem. *Let N be a closed topological manifold which is a mod 2 homology n -sphere, and T be a fixed point free involution on N . Let $f: N \rightarrow M$ be a continuous map of N to a compact topological m -manifold M (with or without boundary). Assume that $n \geq m$ and $f_*: H_n(N; \mathbb{Z}_2) \rightarrow H_n(M; \mathbb{Z}_2)$ is trivial. Then the covering dimension of $A(f) = \{y \in N \mid f(y) = f(Ty)\}$ is at least $n - m$.*

Let π denote the cyclic group of order 2 generated by T . Denote by N_π the orbit space of N , and by $N \times_\pi M^2$ the orbit space of $N \times M^2$ on which π acts by $T(y, x, x^1) = (Ty, x', x)$ ($y \in N, x, x' \in M$). Then $N \times_\pi M^2$ and $N_\pi \times M$ are topological manifolds, and $N_\pi \times M$ is embedded in $N \times_\pi M^2$ by the diagonal map $d: M \rightarrow M^2$. Assuming M is a closed manifold, let $\theta_0 \in H^m(N \times_\pi M^2; \mathbb{Z}_2)$ denote the Poincaré dual of $i^*(w)$, where $w \in H_{n+m}(N \times_\pi M; \mathbb{Z}_2)$ is the fundamental class and $i: N_\pi \times M \subset N \times_\pi M^2$. Define $s: N_\pi \rightarrow N \times_\pi M^2$ by $s(y) = (y, f(y), f(Ty))$ ($y \in N$). Then Conner-Floyd [1] and Munkholm [4] proved their theorems by showing

$s^*(\theta_0) \neq 0$. We also follow this principle (see Lemma 6). However our method of proving $s^*(\theta_0) \neq 0$ is different from theirs, and is purely homological.

Let N be a sufficiently large dimensional sphere, and $T: N \rightarrow N$ be the antipodal map. Assume that M is triangulable. Then Haefliger [2] proved a formula giving θ_0 in terms of cohomology classes of M . We shall show that the formula still holds for our N , T and M , and we shall use the formula to prove $s^*(\theta_0) \neq 0$.

The method can be also applied to obtain the Borsuk-Ulam type theorem for a fixed point free homeomorphism of period p on a mod p homology sphere (p : odd prime), and a theorem including the result in [5] will be proved (see Theorem 8 in §9).

1. Generalization of Eilenberg-Zilber theorem

Throughout §1–§3, a principal ideal domain R is fixed, and chain complexes over R are considered. Thus, the singular complex of a topological space X with coefficients in R is denoted by simply $S(X) = \{S_q(X)\}$, and the tensor product \otimes_R is denoted by simply \otimes .

Let E be a Hausdorff space on which there is given a fixed point free involution T , and such that the reduced homology group $\tilde{H}_*(E)$ is trivial. Then we have the following generalization of Eilenberg-Zilber theorem.

Theorem 1. *There exist chain maps*

$$\begin{aligned}\rho &: S(E \times X_1 \times X_2) \rightarrow S(E) \otimes S(X_1) \otimes S(X_2), \\ \rho' &: S(E) \otimes S(X_1) \otimes S(X_2) \rightarrow S(E \times X_1 \times X_2),\end{aligned}$$

defined for each pair (X_1, X_2) of topological spaces, and satisfying the following conditions:

(i) ρ and ρ' are functorial, i.e. for any continuous maps $f_1: X_1 \rightarrow Y_1$ and $f_2: X_2 \rightarrow Y_2$ we have

$$\begin{aligned}(1 \otimes f_{1\#} \otimes f_{2\#}) \circ \rho &= \rho \circ (1 \times f_1 \times f_2)_{\#}, \\ \rho' \circ (1 \otimes f_{1\#} \otimes f_{2\#}) &= (1 \times f_1 \times f_2)_{\#} \circ \rho'.\end{aligned}$$

(ii) ρ and ρ' are equivariant in the sense that

$$\begin{aligned}(T_{\#} \otimes T) \circ \rho &= \rho \circ (T \times T)_{\#} \\ \rho' \circ (T_{\#} \otimes T) &= (T \times T)_{\#} \circ \rho,\end{aligned}$$

where $T: S(X_1) \otimes S(X_2) \rightarrow S(X_2) \otimes S(X_1)$ is given by $T(c_1 \otimes c_2) = (-1)^{\deg c_1 \deg c_2} c_2 \otimes c_1$, and $T \times T: E \times X_1 \times X_2 \rightarrow E \times X_2 \times X_1$ by $(T \times T)(e, x_1, x_2) = (Te, x_2, x_1)$ ($e \in E, x_1 \in X_1, x_2 \in X_2$).

(iii) *There exist a chain homotopy Φ of $\rho' \circ \rho$ to the identity and a chain homotopy Φ' of $\rho \circ \rho'$ to the identity, which are defined for each pair of topological spaces and which are functorial and equivariant in the same sense as in (i), (ii).*

Proof. The proof is done by the method of acyclic models.

Define a homomorphism $\rho_0: S_0(E \times X_1 \times X_2) \rightarrow S_0(E) \otimes S_0(X_1) \otimes S_0(X_2)$ by $\rho_0(e, x_1, x_2) = e \otimes x_1 \otimes x_2$ ($e \in E, x_1 \in X_1, x_2 \in X_2$), and assume inductively that a homomorphism $\rho_r: S_r(E \times X_1 \times X_2) \rightarrow (S(E) \otimes S(X_1) \otimes S(X_2))_r$ has been defined for $r < n$ so that the conditions

- i) $\partial_r \circ \rho_r = \rho_{r-1} \circ \partial_r,$
- ii) $\rho_r \circ (1 \times f_1 \times f_2)_\# = (1 \otimes f_{1\#} \otimes f_{2\#}) \circ \rho_r,$
- iii) $(T_\# \otimes T) \circ \rho_r = \rho_r \circ (T \times T)_\#$

are satisfied. Take a set $\{e_\lambda^n\}_{\lambda \in \Lambda}$ of singular n -simplexes of E such that $\{e_\lambda^n, T_\# e_\lambda^n\}_{\lambda \in \Lambda}$ is a basis of the module $S_n(E)$. For each $\lambda \in \Lambda$, define a singular n -simplex $d_\lambda^n: \Delta^n \rightarrow E \times \Delta^n \times \Delta^n$ by $d_\lambda^n(z) = (e_\lambda^n(z), z, z)$ ($z \in \Delta^n$). It holds that $\partial_{n-1} \rho_{n-1} \partial_n(d_\lambda^n) = 0$ ($n > 1$) and $\varepsilon \rho_0 \partial_1(d_\lambda^n) = 0$ ($n = 1$) for the augmentation ε . Since the reduced complex of $S(E) \otimes S(\Delta^n) \otimes S(\Delta^n)$ is acyclic, there exists an n -chain $\rho_n(d_\lambda^n)$ of $S(E) \otimes S(\Delta^n) \otimes S(\Delta^n)$ such that $\partial_n \rho_n(d_\lambda^n) = \rho_{n-1} \partial_n(d_\lambda^n)$. The module $S_n(E \times X_1 \times X_2)$ is a free module generated by elements of the form $(1 \times \sigma_1 \times \sigma_2)_\# d_\lambda^n$ or $(T \times \sigma_1 \times \sigma_2)_\# d_\lambda^n$, where $\sigma_i: \Delta^n \rightarrow X_i$ ($i = 1, 2$) is any continuous map. Define a homomorphism $\rho_n: S_n(E \times X_1 \times X_2) \rightarrow S(E) \otimes S(X_1) \otimes S(X_2)_n$ by

$$\begin{aligned} \rho_n((1 \times \sigma_1 \times \sigma_2)_\# d_\lambda^n) &= (1 \otimes \sigma_{1\#} \otimes \sigma_{2\#}) \rho_n(d_\lambda^n), \\ \rho_n((T \times \sigma_1 \times \sigma_2)_\# d_\lambda^n) &= (T_\# \otimes T)(1 \otimes \sigma_{2\#} \otimes \sigma_{1\#}) \rho_n(d_\lambda^n). \end{aligned}$$

Then it is easily checked that the conditions i)—iii) are satisfied for $r = n$. Thus there exists a chain map ρ satisfying the conditions (i) and (ii).

Define a homomorphism $\rho'_0: S_0(E) \otimes S_0(X_1) \otimes S(X_2) \rightarrow S_0(E \times X_1 \times X_2)$ by $\rho'_0(e \otimes x_1 \otimes x_2) = (e, x_1, x_2)$ ($e \in E, x_1 \in X_1, x_2 \in X_2$), and assume inductively that a homomorphism $\rho'_r: (S(E) \otimes S(X_1) \otimes S(X_2))_r \rightarrow S_r(E \times X_1 \times X_2)$ has been defined for $r < n$ so that the conditions

- i)' $\partial_r \circ \rho'_r = \rho'_{r-1} \circ \partial_r,$
- ii)' $(1 \times f_1 \times f_2)_\# \circ \rho'_r = \rho'_r \circ (1 \otimes f_{1\#} \otimes f_{2\#}),$
- iii)' $(T \times T)_\# \circ \rho'_r = \rho'_r \circ (T_\# \otimes T)$

are satisfied. Let $i^k \in S_k(\Delta^k)$ denote the singular simplex given by the identity, and consider $e_\lambda^r \otimes i^s \otimes i^t \in S_r(E) \otimes S_s(\Delta^s) \times S_t(\Delta^t)$ with $r + s + t = n$. It holds that $\partial_{n-1} \rho'_{n-1} \partial_n(e_\lambda^r \otimes i^s \otimes i^t) = 0$ ($n > 1$) and $\varepsilon \rho'_0 \partial_1(e_\lambda^r \otimes i^s \otimes i^t) = 0$ ($n = 1$). Since the reduced complex of $S(E \times \Delta^s \times \Delta^t)$ is acyclic, there exists an n -chain $\rho'_n(e_\lambda^r \otimes i^s \otimes i^t) \in S_n(E \times \Delta^s \times \Delta^t)$ such that $\partial_n \rho'_n(e_\lambda^r \otimes i^s \otimes i^t) = \rho'_{n-1} \partial_n(e_\lambda^r \otimes i^s \otimes i^t)$. The module $(S(E) \otimes S(X_1) \otimes S(X_2))_n$ is a free module generated by elements

of the form $(1 \otimes \sigma_{1\#} \otimes \sigma_{2\#})(e_\lambda^r \otimes i^s \otimes i^t)$ and $(T_\# \otimes \sigma_{1\#} \otimes \sigma_{2\#})(e_\lambda^r \otimes i^s \otimes i^t)$, where $\sigma_1: \Delta^s \rightarrow X$, $\sigma_2: \Delta^t \rightarrow X_2$ are continuous maps. Define a homomorphism $\rho'_n: (S(E) \otimes S(X_1) \otimes S(X_2))_n \rightarrow S_n(E \times X_1 \times X_2)$ by

$$\begin{aligned} & \rho'_n(1 \otimes \sigma_{1\#} \otimes \sigma_{2\#})(e_\lambda^r \otimes i^s \otimes i^t) \\ &= (1 \times \sigma_1 \times \sigma_2)_\# \rho'_n(e_\lambda^r \otimes i^s \otimes i^t), \\ & \rho'_n(T_\# \otimes \sigma_{1\#} \otimes \sigma_{2\#})(e_\lambda^r \otimes i^s \otimes i^t) \\ &= (-1)^{st}(T \times T)_\#(1 \times \sigma_2 \times \sigma_1)_\# \rho'_n(e_\lambda^r \otimes i^s \otimes i^t). \end{aligned}$$

Then it is easily checked that the conditions i)'—iii)' are satisfied for $r=n$. Thus there exists a chain map ρ' satisfying the conditions (i) and (ii).

By the similar method we can construct chain homotopies Φ and Φ' in (iii). This completes the proof of Theorem 1.

The following is obvious from the proof above.

Corollary. *Let E' be a subspace of E which is invariant under T and such that $\tilde{H}_q(E')=0$ for $q < n$. Then ρ and ρ' can be taken in such a way that $\rho_q(S(E' \times X_1 \times X_2)) \subset S(E') \otimes S(X_1) \otimes S(X_2)$ and $\rho'_q(S(E') \otimes S(X_1) \otimes S(X_2)) \subset S(E' \times X_1 \times X_2)$ for $q \leq n$.*

2. Algebraic lemmas

Given a chain complex C , the module $Z(C)$ of cycles of C and the homology module $H(C)$ of C are regarded as chain complexes with trivial boundary operator. Then the inclusion $\xi: Z(C) \rightarrow C$ and the projection $\eta: Z(C) \rightarrow H(C)$ are chain maps.

Let π be a cyclic group of order 2, and T its generator. Let W be a π -free acyclic complex, and define an action of π on the chain complex $C^2 = C \otimes C$ by $T(c_1 \otimes c_2) = (-1)^{\deg c_1 \deg c_2} c_2 \otimes c_1$, $c_1, c_2 \in C$. Consider the diagonal action of π on $W \otimes C^2$, and let $W \otimes_\pi C^2$ denote the quotient complex.

For the homomorphisms

$$\begin{aligned} \xi_*: H(W \otimes_\pi Z(C)^2) &\rightarrow H(W \otimes_\pi C^2), \\ \eta_*: H(W \otimes_\pi Z(C)^2) &\rightarrow H(W \otimes_\pi H(C)^2) \end{aligned}$$

induced by ξ and η , we have

Lemma 1. *If C is a free chain complex such that $H(C)$ is free, then $\xi_* \circ \eta_*^{-1}: H(W \otimes_\pi H(C)^2) \rightarrow H(W \otimes_\pi C^2)$ is well defined and is an isomorphism.*

Proof. There exist chain maps $\eta': H(C) \rightarrow Z(C)$ and $\zeta': C \rightarrow H(C)$ such that $\eta \circ \eta' = 1$, $\zeta' \circ \xi = \eta$. Put $\zeta = \xi \circ \eta': H(C) \rightarrow C$. Then $\zeta_*: H(C) \rightarrow H(C)$ is the identity, and hence ζ is a chain equivalence. Therefore, by a lemma due to

Steenrod (see [8], p. 125), $1 \otimes \zeta^2: W \otimes_{\pi} H(C)^2 \rightarrow W \otimes_{\pi} C^2$ is a chain equivalence, and we have

$$\zeta_*: H(W \otimes_{\pi} H(C)^2) \cong H(W \otimes_{\pi} C^2).$$

Since ζ'_* is the inverse of ζ_* , it follows from $\zeta'_* \circ \xi_* = \eta_*$ that $\xi_* = \zeta_* \circ \eta_*$. Since $\eta_* \circ \eta'_* = 1$, η_* is surjective. Thus we have $\zeta_* = \xi_* \circ \eta_*^{-1}$ which completes the proof.

Denote by C^* the cochain complex dual to a chain complex C . We regard the module $Z(C^*)$ of cocycles of C^* and the cohomology module $H(C^*)$ as cochain complexes with trivial coboundary operator.

Define an action of π on the cochain complex $C^{*2} = C^* \otimes C^*$ by $T(u_1 \otimes u_2) = (-1)^{\deg u_1 \deg u_2} u_2 \otimes u_1$ ($u_1, u_2 \in C^*$), and consider the cochain complex $\text{Hom}_{\pi}(W, C^{*2})$ consisting of equivariant homomorphisms of W to C^{*2} . The inclusion $\xi: Z(C^*) \rightarrow C^*$ and the projection $\eta: Z(C^*) \rightarrow H(C^*)$ induces homomorphisms

$$\begin{aligned} \xi_*: H(\text{Hom}_{\pi}(W, Z(C^*)^2)) &\rightarrow H(\text{Hom}_{\pi}(W, C^{*2})), \\ \eta_*: H(\text{Hom}_{\pi}(W, Z(C^*)^2)) &\rightarrow H(\text{Hom}_{\pi}(W, H(C^*)^2)). \end{aligned}$$

Let $\mu: C^{*2} \rightarrow C^{2*}$ denote the canonical cochain map defined by

$$\langle \mu(u_1 \otimes u_2), c_1 \otimes c_2 \rangle = u(c_1)u_2(c_2) \quad (u_1, u_2 \in C^*, c_1, c_2 \in C).$$

The cochain map dual to $T: C^2 \rightarrow C^2$ defines an action of π on C^{2*} . Then μ is equivariant, and so it induces a homomorphism

$$\mu_*: H(\text{Hom}_{\pi}(W, C^{*2})) \rightarrow H(\text{Hom}_{\pi}(W, C^{2*})).$$

Lemma 2. *Let C be a free non-negative chain complex such that $H(C)$ is of finite type and is free. Then $\xi_* \circ \eta_*^{-1}: H(\text{Hom}_{\pi}(W, H(C^*)^2)) \rightarrow H(\text{Hom}_{\pi}(W, C^{*2}))$ is well defined, and both μ^* and $\xi_* \circ \eta_*^{-1}$ are isomorphisms.*

Proof. There is a free non-negative chain complex C' of finite type such that C and C' are chain equivalent (see [7], p. 246). Let $\varphi: C \rightarrow C'$ be a chain equivalence, and consider the following commutative diagram

$$\begin{array}{ccc} H(\text{Hom}_{\pi}(W, H(C'^*)^2)) & \xrightarrow{\varphi_*} & H(\text{Hom}_{\pi}(W, H(C^*)^2)) \\ \uparrow \eta_* & & \uparrow \eta_* \\ H(\text{Hom}_{\pi}(W, Z(C'^*)^2)) & \xrightarrow{\varphi_*} & H(\text{Hom}_{\pi}(W, Z(C^*)^2)) \\ \downarrow \xi_* & & \downarrow \xi_* \\ H(\text{Hom}_{\pi}(W, C'^{*2})) & \xrightarrow{\varphi_*} & H(\text{Hom}_{\pi}(W, C^{*2})) \\ \downarrow \mu_* & & \downarrow \mu_* \\ H(\text{Hom}_{\pi}(W, C'^{2*})) & \xrightarrow{\varphi_*} & H(\text{Hom}_{\pi}(W, C^{2*})) \end{array}$$

Since C'^* and $H(C'^*)$ are free, the argument similar to the proof of Lemma 1 shows that $\xi_* \circ \eta_*^{-1}$ in the left side is well defined and is an isomorphism. Since C' is of finite type and is free, $\mu: C'^{*2} \rightarrow C'^{2*}$ is an isomorphism, and so is μ_* in the left side. Since φ is a chain equivalence, it follows that φ_* in the 3-rd and the 4-th rows are isomorphisms. Obviously φ_* in the 1-st row is also an isomorphism. Thus we obtain the desired result.

By the definitions, $H(W \otimes_{\pi} H(C)^2)$ is the homology group $H(\pi; H(C)^2)$ of the group π with coefficients in the module $H(C)^2$ on which π acts by $T(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$ ($a, b \in H(C)$), and $H(\text{Hom}_{\pi}(W, H(C^*)^2))$ is the cohomology group $H(\pi; H(C^*)^2)$ of the group π with coefficients in the module $H(C^*)^2$ on which π acts by $T(\alpha \otimes \beta) = (-1)^{\deg \alpha \deg \beta} \beta \otimes \alpha$ ($\alpha, \beta \in H(C^*)$).

3. Homology and cohomology of $E \times_{\pi} X^2$

Given a topological space Y on which π acts, we denote by Y_{π} the orbit space. For a topological space X , consider the space $E \times X^2$ on which π acts by $T(e, x, x') = (Te, x', x)$ ($e \in E, x, x' \in X$). We write $(E \times X^2)_{\pi} = E \times_{\pi} X^2$.

For the singular homology group and the singular cohomology group of $E \times_{\pi} X^2$, we have

Theorem 2. (i) *There exists a functorial isomorphism*

$$\kappa: H_*(\pi; H_*(X)^2) \cong H_*(E \times_{\pi} X^2),$$

defined for each topological space X such that $H_(X)$ is free.*

(ii) *There exists a functorial isomorphism*

$$\kappa: H^*(\pi; H^*(X)^2) \cong H^*(E \times_{\pi} X^2),$$

defined for each topological space X such that $H_(X)$ is of finite type and is free.*

Proof. (i) The action of π on $E \times X^2$ makes the singular complex $S(E \times X^2)$ a π -complex. Let $S(E \times X^2)/\pi$ denote the quotient complex. Since the projection $p: E \times X^2 \rightarrow E \times_{\pi} X^2$ is a fibering with discrete fiber, it follows that $p_{\sharp}: S(E \times X^2) \rightarrow S(E \times_{\pi} X^2)$ induces an isomorphism

$$S(E \times X^2)/\pi \cong S(E \times_{\pi} X^2).$$

Define an action of π on $S(E) \otimes S(X)^2$ by $T(c \otimes c_1 \otimes c_2) = (-1)^{\deg c_1 \deg c_2} T_{\sharp}(c) \otimes c_2 \otimes c_1$ ($c \in S(E)$, $c_1, c_2 \in S(X)$). Then it follows that ρ' in Theorem 1 induces a chain equivalence

$$S(E) \otimes S(X)^2 \rightarrow S(E \times_{\pi} X^2)/\pi.$$

Therefore an isomorphism

$$\chi: H(S(E) \otimes_{\pi} S(X)^2) \cong H_*(E \times_{\pi} X^2)$$

is induced by the chain map $p_{\sharp} \circ \rho'$. Since p_{\sharp} and ρ' are functorial, so is χ .

Since $S(E)$ is a π -free acyclic complex, by Lemma 1 we have

$$\xi_* \circ \eta_*^{-1}: H(S(E) \otimes_{\pi} H(X)^2) \cong H(S(E) \otimes_{\pi} S(X)^2).$$

Obviously ξ_* and η_* are functorial. Therefore the desired isomorphism κ is given by $\kappa = \chi \circ \xi_* \circ \eta_*^{-1}$.

(ii) The cochain complex $\text{Hom}_{\pi}(S(E), S(X)^{2*})$ is canonically isomorphic with the cochain complex $(S(E) \otimes_{\pi} S(X)^2)^*$, the above proof of (i) shows that an isomorphism

$$\chi': H(\text{Hom}_{\pi}(S(E), S(X)^{2*})) \cong H^*(E \times_{\pi} X^2)$$

is induced by the chain map $p_{\sharp} \circ \rho'$. On the other hand, by Lemma 2 we have

$$\mu_* \circ \xi_* \circ \eta_*^{-1}: H(\text{Hom}_{\pi}(S(E), H^*(X)^2)) \cong H(\text{Hom}_{\pi}(S(E), S(X)^{2*})).$$

Therefore the desired isomorphism κ is given by $\kappa = \chi \circ \xi_* \circ \eta_*^{-1}$ with $\chi = \chi' \circ \mu_*$. This completes the proof of Theorem 2.

Define a pairing of $H^*(X)^2$ and $H^*(X)^2$ to $H^*(X)^2$ by

$$\begin{aligned} (\alpha \otimes \beta) \cdot (\gamma \otimes \delta) &= (-1)^{\deg \beta \deg \gamma} (\alpha \smile \gamma) \otimes (\beta \smile \delta) \\ &(\alpha, \beta, \gamma, \delta \in H^*(X)). \end{aligned}$$

Since this pairing is equivariant with respect to the action on $H^*(X)^2$, it gives rise to a cup product

$$\smile: H^*(\pi; H^*(X)^2) \otimes H^*(\pi; H^*(X)^2) \rightarrow H^*(\pi; H^*(X)^2).$$

Similarly, an equivariant pairing of $H^*(X)^2$ and $H_*(X)^2$ to $H_*(X)^2$ defined by

$$\begin{aligned} (\alpha \otimes \beta) \cdot (a \otimes b) &= (-1)^{\deg \alpha (\deg b - \deg \beta)} (\alpha \frown a) \otimes (\beta \frown b) \\ &(\alpha, \beta \in H^*(X), a, b \in H_*(X)) \end{aligned}$$

gives rise to a cap product

$$\frown: H^*(\pi; H^*(X)^2) \otimes H_*(\pi; H_*(X)^2) \rightarrow H_*(\pi; H_*(X)^2).$$

Theorem 3. *The isomorphisms κ in Theorem 2 preserve the cup products and the cap products, i.e. the following diagrams are commutative.*

$$\begin{array}{ccc} H^*(\pi; H^*(X)^2) \otimes H^*(\pi; H^*(X)^2) & \xrightarrow{\smile} & H^*(\pi; H^*(X)^2) \\ \downarrow \kappa \otimes \kappa & & \downarrow \kappa \\ H^*(E \times_{\pi} X^2) \otimes H^*(E \times_{\pi} X^2) & \xrightarrow{\smile} & H^*(E \times_{\pi} X^2), \end{array}$$

$$\begin{array}{ccc}
H^*(\pi; H^*(X)^2) \otimes H_*(\pi; H_*(X)^2) & \xrightarrow{\quad} & H_*(\pi; H_*(X)^2) \\
\downarrow \kappa \otimes \kappa & & \downarrow \kappa \\
H^*(E \times X^2) \otimes H_*(E \times X^2) & \xrightarrow{\quad} & H_*(E \times X^2).
\end{array}$$

Proof. For $C=S(X)$ and $Z(S(X))$, a cup product

$$\smile: \text{Hom}_\pi(S(E), C^{*2}) \otimes \text{Hom}_\pi(S(E), C^{*2}) \rightarrow \text{Hom}_\pi(S(E), C^{*2})$$

and a cap product

$$\frown: \text{Hom}_\pi(S(E), C^{2*}) \otimes (S(E) \otimes_\pi C^2) \rightarrow S(E) \otimes_\pi C^2$$

are defined similarly to the above, by using of the cup product and the cap product for cochains and chains. Then it is obvious that the homomorphisms ξ_* and η_* in the proof of Theorem 2 preserve the cup products and the cap products. Therefore it suffices to prove that the homomorphisms χ in the proof of Theorem 2 preserve the cup products and the cap products.

For any topological space Y , let $\Delta: S(Y) \rightarrow S(Y)^2$ denote the diagonal approximation (see [7], p. 250). Consider a diagram

$$\begin{array}{ccccc}
S(E) \otimes S(X)^2 & \xrightarrow{\Delta \otimes \Delta^2} & S(E)^2 \otimes (S(X)^2)^2 & \xrightarrow{\tau} & (S(E) \otimes S(X)^2)^2 \\
\downarrow \rho' & & & & \downarrow \rho'^2 \\
S(E \times X^2) & & \xrightarrow{\Delta} & & S(E \times X^2)^2 \\
\downarrow p_\# & & \xrightarrow{\Delta} & & \downarrow p_\#^2 \\
S(E \times_\pi X^2) & & & & S(E \times_\pi X^2)^2,
\end{array}$$

where τ is the appropriate chain map shuffling factors. Since Δ is functorial, the lower rectangle is commutative. Regard $S(E \times X^2)^2$ and $(S(E) \otimes S(X)^2)^2$ as π -complexes by the diagonal action of the actions of $S(E \times X^2)$ and $S(E) \otimes S(X)^2$ respectively. Then it follows that the maps in the 1-st and the 2-nd rows are equivariant. Furthermore the argument similar to the proof of Theorem 1 shows that there exists a chain homotopy of $\Delta \circ \rho'$ to $\rho'^2 \circ \tau \circ (\Delta \otimes \Delta^2)$ which is equivariant and functorial. Therefore we have a diagram

$$\begin{array}{ccc}
S(E) \otimes_\pi S(X)^2 & \xrightarrow{\tau \circ (\Delta \otimes \Delta^2)} & (S(E) \otimes_\pi S(X)^2)^2 \\
\downarrow p_\# \circ \rho' & & \downarrow (p_\# \circ \rho')^2 \\
S(E \times_\pi X^2) & \xrightarrow{\Delta} & S(E \times_\pi X^2)^2
\end{array}$$

which is commutative up to chain homotopy.

Recall now the definition of cup product (cap product) in terms of Δ and cross (slant) product. Then the above diagram yields the desired property. This completes the proof of Theorem 3.

4. Steenrod theorem

In §4–§8, we assume that the ground ring R is \mathbb{Z}_2 , the field of integers mod 2. We assume also that $H_*(X)$ is of finite type.

As is well known, a π -free acyclic complex W can be constructed as follows: For each $i \geq 0$, W has one cell e_i and its transform Te_i , and $\partial(e_i) = e_{i-1} + Te_{i-1}$ ($i > 0$). Moreover there is a diagonal approximation $d_\# : W \rightarrow W \otimes W$ which is given by

$$d_\#(e_i) = \sum_{j=0}^{\lfloor i/2 \rfloor} (e_{2j} \otimes e_{i-2j} + e_{2j+1} \otimes Te_{i-2j-1}).$$

Therefore we can determine the structure of $H_*(\pi; H_*(X)^2)$ and $H^*(\pi; H^*(X)^2)$, and hence by Theorems 2 and 3 the structure of $H_*(E \times_\pi X^2)$ and $H_*(E \times_\pi X^2)$ as soon as we know the structure of $H_*(X)$. To state the result, we shall first prepare some notations.

For an element $a \in H_*(X)$, let $Q_i(a) \in H_*(\pi; H_*(X)^2)$ ($i \geq 0$) denote the element represented by the cycle $e_i \otimes a \otimes a \in W \otimes H_*(X)^2$. Given an element $\alpha \in H^*(X)$ a cocycle $u_i(\alpha) \in \text{Hom}_\pi(W, H^*(X)^2)$ is defined by $\langle u_i(\alpha), e_j \rangle = \alpha \otimes \alpha$ ($i=j$), $=0$ ($i \neq j$). Let $Q_i(\alpha) \in H^*(\pi; H^*(X)^2)$ denote the element represented by $u_i(\alpha)$.

For two elements $a, b \in H_*(X)$, let $Q(a, b) \in H_*(\pi; H_*(X)^2)$ denote the element represented by the cycle $e_0 \otimes a \otimes b \in W \otimes H_*(X)^2$. Given two elements $\alpha, \beta \in H^*(X)$, a cocycle $u(\alpha, \beta) \in \text{Hom}_\pi(W, H^*(X)^2)$ is defined by $\langle u(\alpha, \beta), e_j \rangle = \alpha \otimes \beta + \beta \otimes \alpha$ ($j=0$), $=0$ ($j \neq 0$). Let $Q(\alpha, \beta) \in H^*(\pi; H^*(X)^2)$ denote the element represented by $u(\alpha, \beta)$.

We shall put

$$\begin{aligned} P_i(a) &= \kappa(Q_i(a)), & P(a, b) &= \kappa(Q(a, b)), \\ P_i(\alpha) &= \kappa(Q_i(\alpha)), & P(\alpha, \beta) &= \kappa(Q(\alpha, \beta)). \end{aligned}$$

Obviously we have $P(a, b) = P(b, a)$, $P(a, a) = P_0(a)$, $P(\alpha, \beta) = P(\beta, \alpha)$ and $P(\alpha, \alpha) = 0$.

The following theorem is proved easily.

Theorem 4. (i) If $\{a_j \mid j \in J\}$ is an ordered basis of the module $H_*(X)$, then $\{P_i(a_j), P(a_j, a_k) \mid i \geq 0, j, k \in J, j > k\}$ is a basis of the module $H_*(E \times_\pi X^2)$.

Similarly, if $\{\alpha_j \mid j \in J\}$ is an ordered basis of the module $H^*(X)$, then $\{P_i(\alpha_j), P(\alpha_j, \alpha_k) \mid i \geq 0, j, k \in J, j > k\}$ is a basis of the module $H^*(E \times_\pi X^2)$.

(ii) For $\alpha, \beta, \alpha', \beta' \in H^*(X)$ and $a, b \in H_*(X)$, we have

$$\begin{aligned} P_i(\alpha) \smile P_j(\beta) &= P_{i+j}(\alpha \smile \beta); \\ P_i(\alpha) \smile P(\alpha', \beta') &= \begin{cases} P(\alpha \smile \alpha', \alpha \smile \beta') & \text{if } i=0, \\ 0 & \text{if } i>0; \end{cases} \end{aligned}$$

$$P(\alpha, \beta) \smile P(\alpha', \beta') = P(\alpha \smile \alpha', \beta \smile \beta') + P(\alpha \smile \beta', \beta \smile \alpha'),$$

$$P_j(\alpha) \frown P_i(a) = \begin{cases} P_{i-j}(\alpha \frown a) & \text{if } i \geq j, \\ 0 & \text{if } i < j; \end{cases}$$

$$P(\alpha, \beta) \frown P_i(a) = 0;$$

$$P_j(\alpha) \frown P(a, b) = \begin{cases} P(\alpha \frown a, \alpha \frown b) & \text{if } j=0, \\ 0 & \text{if } j>0; \end{cases}$$

$$P(\alpha, \beta) \frown P(a, b) = P(\alpha \frown a, \beta \frown b) + P(\beta \frown a, \alpha \frown b),$$

where it is to be understood that $\alpha \frown a = 0$ if $\deg \alpha > \deg a$.

(iii) If $\{a_j\}$ and $\{\alpha_j\}$ are dual bases, then so are $\{P(a_j), P(a_j, a_k)\}$ and $\{P_i(\alpha_j), P(\alpha_j, \alpha_k)\}$.

Define a continuous map $d: E \times X \rightarrow E \times X^2$ by $d(y, x) = (y, x, x)$ ($y \in E, x \in X$). Then d induces a homomorphism

$$d^*: H^*(E \times X^2) \rightarrow H^*(E_\pi \times X).$$

We have the following theorem due to Steenrod (see [8], p. 103).

Theorem 5. For $\alpha \in H^q(X)$ we have

$$d^*P_0(\alpha) = \sum_{k=0}^q \omega^k \times Sq^{q-k}\alpha,$$

where $\omega^k \in H^k(E_\pi)$ is the generator, and $Sq^i: H^q(X) \rightarrow H^{q+i}(X)$ is the squaring operation.

Proof. Let $\lambda: S(E) \otimes S(X) \rightarrow S(X)^2$ be a functorial equivariant chain map defined for each topological space X , and $\varphi: S(E \times X) \rightarrow S(E) \otimes S(X)$ be a chain equivalence in the Eilenberg-Zilber theorem. Let $u \in S(X)^*$ be a cocycle representing α . Then it follows from the definition of Sq^i that $\varphi^* \lambda^* \mu(u \otimes u) \in (S(E) \otimes S(X))^*$ is an equivariant cocycle representing $\sum_{k=0}^q \omega^k \times Sq^{q-k}\alpha \in H^{2q}(E_\pi \times X)$, where $\mu: S(X)^{2*} \rightarrow S(X)^{2*}$ is the canonical cochain map. Consider the composition of chain maps

$$\begin{aligned} S(E) \otimes S(X) &\xrightarrow{\varphi'} S(E \times X) \xrightarrow{d_*} S(E \times X^2) \\ &\xrightarrow{\rho} S(E) \otimes S(X)^2 \xrightarrow{\varepsilon \otimes 1} S(X)^2, \end{aligned}$$

where φ' is an inverse of φ and ε is the augmentation. Since each map is functorial and equivariant, we can take the composition as λ . It is easily seen that $\rho^*(\varepsilon \otimes 1)^* \mu(u \times u) \in S(E \times X^2)^*$ is an equivariant cocycle representing $P_0(\alpha)$. Therefore we have the desired result.

Corollary. For $\alpha \in H^q(X)$ we have

$$d^*P_i(\alpha) = \sum_{k=0}^q \omega^{k+i} \times Sq^{q-k}\alpha.$$

Proof. Since $P_i(\alpha) = P_i(1) \smile P_0(\alpha)$ by Theorem 4, we have $d^*P_i(\alpha) = d^*P_i(1) \smile d^*P_0(\alpha)$. Let P be a single point, and consider the commutative diagram

$$\begin{array}{ccccc} H^*(\pi; H^*(P)^2) & \xrightarrow{\kappa} & H^*(E \times_{\pi} P^2) & \xrightarrow{d^*} & H^*(E_{\pi} \times P) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(\pi; H^*(X)^2) & \xrightarrow{\kappa} & H^*(E \times_{\pi} X^2) & \xrightarrow{d^*} & H^*(E_{\pi} \times X), \end{array}$$

where the vertical maps are induced by the map $X \rightarrow P$. Then it follows that $d^*P_i(1) = \omega^i \times 1$. Therefore the corollary follows from the theorem.

5. Homology of $N \times_{\pi} X^2$

Let Y be a topological space, and consider the suspension SY of Y . We regard SY as the quotient space of $Y \times [0, 1]$, and identify Y with the subspace $Y \times 1/2$ of SY . If T is an involution on Y , it is extended to an involution T' on SY by putting

$$T'(y, s) = (Y(y), 1-s) \quad (y \in Y, 0 \leq s \leq 1).$$

If T has no fixed point, so does T' .

Let N be a compact Hausdorff space having the mod 2 homology of the n -sphere, and suppose that there is given on N an involution without fixed points. Define now, for each integer $i \geq 0$, a compact Hausdorff space N^i by

$$N^0 = N, \quad N^i = SN^{i-1} \quad (i \geq 1),$$

and let N^{∞} denote the inductive limit of the sequence $N \subset N^1 \subset \cdots \subset N^i \subset \cdots$. Then, by the fact stated above, there exists an involution $T: N^{\infty} \rightarrow N^{\infty}$ without fixed points such that $T(N^i) = N^i$ ($i \geq 0$) and $T|N$ is the given involution. Moreover, since $\tilde{H}_q(N^i) \cong \tilde{H}_{q-i}(N) = 0$ ($i > q - n$), we have $\tilde{H}_q(N^{\infty}) = \lim_{\rightarrow} \tilde{H}_q(N^i) = 0$ for any q . Thus N^{∞} has the properties assumed for E .

Assuming next that X is an arcwise connected topological space, we shall consider the space $N^{\infty} \times_{\pi} X^2$ and its subspace $N \times_{\pi} X^2$.

Theorem 6. The homomorphism $i_*: H_k(N \times_{\pi} X^2) \rightarrow H_k(N^{\infty} \times_{\pi} X^2)$ induced by the inclusion is an isomorphism if $k \leq n$.

Proof. The projection $N^{\infty} \times X^2 \rightarrow N^{\infty}$ to the first factor defines a fibration $p: N^{\infty} \times_{\pi} X^2 \rightarrow N^{\infty}$ with fiber X^2 , and we have $p^{-1}(N^i) = N^i \times_{\pi} X^2$ for each i .

Consider the Serre spectral sequence E for the relative fibration $p: (N^\infty \times_\pi X^2, N \times_\pi X^2) \rightarrow (N_\pi^\infty, N_\pi)$. Then we have

$$E_{p,q}^2 \cong H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\}),$$

and E^∞ is the graded module associated to some filtration of $H_*(N^\infty \times_\pi X^2, N \times_\pi X^2)$. By the properties of homology, it holds that

$$\begin{aligned} H_p(N_\pi^t, N_\pi^{t-1}; \{H_q(X^2)\}) &\cong H_p(CN^{i-1}, N^{i-1}, H_q(X^2)) \\ &\cong \tilde{H}_{p-1}(N^{i-1}; H_q(X^2)) \cong \tilde{H}_{p-i}(N) \otimes H_q(X^2) \\ &\cong \begin{cases} H_q(X^2) & \text{if } p-i=n, \\ 0 & \text{if } p-i \neq n, \end{cases} \end{aligned}$$

where $i \geq 1$ and CN^{i-1} denote the cone over N^{i-1} . Therefore the homomorphism $H_p(N_\pi^{t-1}, N_\pi; \{H_q(X^2)\}) \rightarrow H_p(N_\pi^t, N_\pi; \{H_q(X^2)\})$ is injective if $p-i \neq n-1$, and is surjective if $p-i \neq n$. Hence $H_p(N_\pi^t, N_\pi; \{H_q(X^2)\}) \cong H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\})$ if $p-i \leq n-1$. In particular, $H_p(N_\pi, N_\pi; \{H_q(X^2)\}) \rightarrow H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\})$ is surjective if $p \leq n$, and so we have

$$E_{p,q}^2 = 0 \quad (p \leq n).$$

The usual technique in spectral sequence proves now that $H_k(N^\infty \times_\pi X^2, N \times_\pi X^2) = 0$ for $k \leq n$. Thus $i_*: H_k(N \times_\pi X^2) \rightarrow H_k(N^\infty \times_\pi X^2)$ is bijective if $k \leq n-1$, and is surjective if $k=n$.

We shall next prove that $i_*: H_n(N \times_\pi X^2) \rightarrow H_n(N^\infty \times_\pi X^2)$ is injective.

Since $H_{n+1}(N_\pi) = 0$, the homomorphism $H_{n+1}(N_\pi^\infty) \rightarrow H_{n+1}(N_\pi^\infty, N_\pi)$ is injective. On the other hand, $Z_2 \cong H_{n+1}(N_\pi^1, N_\pi) \rightarrow H_{n+1}(N_\pi^\infty, N_\pi)$ is surjective. Therefore we have $H_{n+1}(N_\pi^\infty) \cong H_{n+1}(N_\pi^\infty, N_\pi)$.

Consider the Serre spectral sequence $'E$ of the fibration $p: N^\infty \times_\pi X^2 \rightarrow N_\pi^\infty$. Then we have $'E_{p,q}^2 \cong H_p(N_\pi^\infty; \{H_q(X^2)\})$, and $'E^\infty$ is the graded module associated with some filtration of $H_*(N^\infty \times_\pi X^2)$. Since $H_*(N^\infty \times_\pi X^2) \cong H_*(N_\pi^\infty; \{H_*(X^2)\})$ by Theorem 2, the usual technique in spectral sequence proves that $'E_{p,q}^2 = 'E_{p,q}^\infty$.

Consider now the commutative diagram

$$\begin{array}{ccc} H_{n+1}(N^\infty \times_\pi X^2) & \longrightarrow & 'E_{n+1,0}^2 = H_{n+1}(N_\pi^\infty) \\ \downarrow & & \downarrow \cong \\ H_{n+1}(N^\infty \times_\pi X^2, N \times_\pi X^2) = E_{n+1,0}^2 & = & H_{n+1}(N_\pi^\infty, N_\pi) \end{array}$$

Then it follows that the map in the left is surjective. Therefore $i_*: H_n(N \times_\pi X^2) \rightarrow H_n(N^\infty \times_\pi X^2)$ is injective. This completes the proof of Theorem 6.

Lemma 3. Let $i \leq n$ and $a \in H_*(X)$. Then $P_i(a)$ is in the image of the homomorphism $i_*: H_*(N \times_\pi X^2) \rightarrow H_*(N^\infty \times_\pi X^2)$.

Proof. Since $i \leq n$, we have $H_i(N_\pi^\infty, N_\pi; \{H_*(X^2)\}) = 0$. Hence the homomorphism $H_i(N_\pi; \{H_*(X^2)\}) \rightarrow H_i(N_\pi^\infty; \{H_*(X^2)\})$ is surjective. This shows that $Q_i(a)$ is represented by a cycle of $S(N) \otimes S(X)^2$. Since $\tilde{H}_i(N) = 0$ ($i < n$) and $T(N) = N$, it follows from Corollary to Theorem 1 that $P_i(a)$ is represented by a cycle of $S(N \times X^2)$. Therefore we get the desired result.

6. The element θ

Let N be a closed topological manifold having the mod 2 homology of the n -sphere, and let M be a connected closed topological manifold of dimension m . Suppose that there is given on N an involution without fixed points. Then $N \times M^2$ is a connected closed topological manifold of dimension $n + 2m$. Let $\mu \in H_m(M)$ and $\lambda \in H_{2m+n}(N \times M^2)_\pi$ denote the mod 2 fundamental class of M and $N \times M^2$ respectively.

By Lemma 3, $P_n(\mu) \in H_{n+2m}(N \times M^2)_\pi$ is in the image of the homomorphism $i_*: H_{n+2m}(N \times M^2)_\pi \rightarrow H_{n+2m}(N \times M^2)_\pi$. Therefore we have

$$P_n(\mu) = i_*(\lambda).$$

Define a continuous map $d_0: N \times M \rightarrow N \times M^2$ by $d_0(y, x) = (y, x, x)$ ($y \in N, x \in M$). Then d_0 induces a homomorphism $d_{0*}: H_*(N_\pi \times M) \rightarrow H_*(N \times M^2)_\pi$. Define

$$\theta_0 \in H^m(N \times M^2)_\pi$$

to be the Poincaré dual of $d_{0*}(v_n \times \mu)$, where $v_n \in H_n(N_\pi)$ is the generator. We have

$$\theta_0 \frown \lambda = d_{0*}(v_n \times \mu).$$

Assume now that $m \leq n$. Since $i^*: H^k(N \times M^2)_\pi \cong H^k(N \times M^2)_\pi$ for $k \leq n$ by Theorem 6, there exists a unique element $\theta \in H^m(N \times M^2)_\pi$ such that

$$i^*(\theta) = \theta_0.$$

For the homomorphism $d_*: H_*(N \times M)_\pi \rightarrow H_*(N \times M^2)_\pi$ induced by the 'diagonal' map, we have

$$d_*(i_*(v_n) \times \mu) = \theta \frown P_n(\mu).$$

In fact,

$$\begin{aligned} d_*(i_*(v_n) \times \mu) &= d_*i_*(v_n \times \mu) \\ &= i_*d_{0*}(v_n \times \mu) = i_*(\theta_0 \frown \lambda) \\ &= i_*(i^*(\theta) \frown \lambda) = \theta \frown i_*(\lambda) \\ &= \theta \frown P_n(\mu). \end{aligned}$$

Let $U_i \in H^i(M)$ denote the Wu class, i.e. the element defined by

$$U_i \cap \mu = Sq^{i*}(\mu),$$

where $Sq^{i*}: H_m(M) \rightarrow H_{m-i}(M)$ is the transpose of $Sq^i: H^{m-i}(M) \rightarrow H^m(M)$.

Theorem 7. *If $m \leq n$, we have*

$$\theta = \sum_{i=0}^{\lfloor m/2 \rfloor} P_{m-2i}(U_i) + \delta,$$

where δ is a linear combination of elements of the type $P(\alpha, \beta)$.

Proof. For any $\alpha \in H^q(M)$ with $2q \geq m \geq q$, we have

$$\begin{aligned} & \langle P_{n+m-2q}(\alpha), d_*(i_*(v_n) \times \mu) \rangle \\ &= \langle P_{n+m-2q}(\alpha), \theta \frown P_n(\mu) \rangle \\ &= \langle P_{n+m-2q}(\alpha) \smile \theta, P_n(\mu) \rangle \\ &= \langle \theta, P_{n+m-2q}(\alpha) \frown P_n(\mu) \rangle \\ &= \langle \theta, P_{2q-m}(\alpha \frown \mu) \rangle \quad (\text{by (ii) of Theorem 4}). \end{aligned}$$

We have also

$$\begin{aligned} & \langle P_{n+m-2q}(\alpha), d_*(i_*(v_n) \times \mu) \rangle \\ &= \langle d^* P_{n+m-2q}(\alpha), i^*(v_n) \times \mu \rangle \\ &= \left\langle \sum_{i=0}^q \omega^{n+m-2q+i} \times Sq^{q-i}(\alpha), i_*(v_n) \times \mu \right\rangle \\ & \quad (\text{by Corollary of Theorem 5}) \\ &= \langle \omega^n \times Sq^{m-q}(\alpha), i_*(v_n) \times \mu \rangle \\ &= \langle Sq^{m-q}(\alpha), \mu \rangle = \langle \alpha, U_{m-q} \frown \mu \rangle \\ &= \langle U_{m-q}, \alpha \frown \mu \rangle \\ &= \left\langle \sum_{i=0}^{\lfloor m/2 \rfloor} P_{m-2i}(U_i), P_{2q-m}(\alpha \frown \mu) \right\rangle \quad (\text{by (ii) of Theorem 4}). \end{aligned}$$

Therefore we get the desired result by (iii) of Theorem 4.

7. Proof of the main theorem

In this section we shall prove the main theorem.

For a continuous map $f: N \rightarrow M$, a continuous $s: N_\pi \rightarrow N \times_\pi M^2$ can be defined by

$$s(y) = (y, f(y), f(Ty)) \quad (y \in N).$$

For the homomorphism $s^*: H^m(N \times_\pi M^2) \rightarrow H^m(N_\pi)$, we have

Lemma 4. *If $m \leq n$ and $f_*: H_n(N) \rightarrow H_n(M)$ is trivial, it holds that $s^*(\theta_0) \neq 0$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} H^m(N^\infty \times_\pi M^2) & \xrightarrow{i^*} & H^m(N \times_\pi M^2) \\ \downarrow (1 \times f^2) & & \downarrow s^* \\ H^m(N^\infty \times_\pi N^2) & \xrightarrow{k^*} & H^m(N_\pi), \end{array}$$

where $k: N_\pi \rightarrow N^\infty \times_\pi N^2$ is given by $k(y) = (y, y, Ty)$ ($y \in N$). Therefore, by Theorem 7 we have

$$\begin{aligned} s^*(\theta_0) &= s^*i^*(\theta) = k^*(1 \times f^2)^*(\theta) \\ &= k^*(1 \times f^2)^*\left(\sum_{i=0}^{[m/2]} P_{m-2i}(U_i) + \delta\right). \end{aligned}$$

From this and the assumption it follows that

$$s^*(\theta_0) = k^*(P_m(1)).$$

If P is a single point and $g: N \rightarrow P$ is the map, the diagram

$$\begin{array}{ccc} H^m(N_\pi^\infty) & \xrightarrow{p^*} & H^m(N^\infty \times_\pi P^2) \\ \downarrow i^* & \searrow p^* & \downarrow (1 \times g^2)^* \\ H^m(N_\pi) & \xleftarrow{k^*} & H^m(N^\infty \times_\pi N^2) \end{array}$$

is commutative. Therefore we have

$$k^*(P_m(1)) = k^*p^*(\omega^m) = i^*(\omega^m) \neq 0,$$

which completes the proof of Lemma 4.

For a continuous map $f: N \rightarrow M$, put

$$A(f) = \{y \in N \mid f(y) = f(Ty)\}.$$

Lemma 5. *If $s^*(\theta_0) \neq 0$, then the covering dimension of $A(f)$ is at least $n - m$.*

Proof. By the Thom isomorphism, we have

$$H^0(N \times_\pi M^2) \cong H^{n+2m}((N \times_\pi M^2)^2, (N \times_\pi M^2)^2 - \Delta(N \times_\pi M^2)),$$

where $\Delta: N \times_\pi M^2 \rightarrow (N \times_\pi M^2)^2$ is the diagonal map. Let $\gamma \in H^{n+2m}((N \times_\pi M^2)^2, (N \times_\pi M^2)^2 - \Delta(N \times_\pi M^2))$ be the generator, and put

$$\gamma_1 = \gamma|_{(N \times_\pi M^2)^2},$$

$$\gamma_2 = \gamma|_{d_0(N_\pi \times M) \times (N \times_\pi M^2, N \times_\pi M^2 - d_0(N_\pi \times M))}.$$

Write $B(f)$ for the image of $A(f)$ under the projection $N \rightarrow N_\pi$. Then the following commutative diagram holds:

$$\begin{array}{ccc}
 H_{n+m}(d_0(N_\pi \times M)) & \xrightarrow{j_*} & H_{n+m}(N_\pi \times M^2) \\
 \downarrow \backslash \gamma_2 & & \downarrow \backslash \gamma_1 \\
 H_m(N_\pi \times M^2, N_\pi \times M^2 - d_0(N_\pi \times M)) & \xrightarrow{j^*} & H_m(N_\pi \times M^2) \\
 \downarrow s^* & & \downarrow s^* \\
 H_m(N, N_\pi - B(f)) & \xrightarrow{j^*} & H_m(N_\pi),
 \end{array}$$

where j are the inclusion maps, and \backslash denotes the slant product (see [7], p. 351). Since $\backslash \gamma_1$ is the inverse of the Poincaré duality isomorphism, the image of the generator of $H_{n+m}(d_0(N_\pi \times M))$ under the composition of j_* and $\backslash \gamma_1$ is θ_0 . Therefore Lemma 4 implies $H_m(N_\pi, N_\pi - B(f)) \neq 0$. Since this shows $H_m(N_\pi, N_\pi - B(f)) \neq 0$, it follows that the Čech cohomology group $\check{H}^{n-m}(B(f))$ is not zero (see Theorem 17 in p. 296, Corollary 8 in p. 334 and Corollary 9 in p. 341 of [7]). Therefore $\dim(B(f)) \geq n-m$, and hence $\dim(A(f)) \geq n-m$. This completes the proof of Lemma 5.

We are now ready for proving the main theorem.

Proof of Main Theorem. By Lemma 4 and Lemma 5, we have the main theorem for a connected closed topological manifold M . From this the result for any compact manifold M is obtained easily (see [4]).

8. Corollaries of the main theorem

The following corollary is obtained immediately from the main theorem.

Corollary 1. *Let N , T and M be the same as in the main theorem, and T' be a fixed point free involution on M .*

(i) *If $n > m$, there exists no continuous map $f: N \rightarrow M$ equivariant with T and T' .*

(ii) *If $n = m$ and $f: N \rightarrow M$ is a continuous map equivariant with T and T' , then $f_*: H_n(N) \rightarrow H_n(M)$ is not trivial.*

The following corollaries are obtained by the same way as in [1], p. 89.

Corollary 2. *Let N be a closed topological manifold which is a mod 2 homology n -sphere. Then any pair of fixed point free involution T_1 and T_2 on N have a co-incidence.*

Corollary 3. *If G is a group acting freely on a closed topological manifold N having the mod 2 homology of the n -sphere. Then any element of G order 2 lies in the center of G .*

REMARK. This corollary was first proved by Milnor [3] in a different method.

9. The corresponding theorem for \mathbb{Z}_p -actions

The main theorem has the following corresponding result for \mathbb{Z}_p -actions on mod p homology spheres, where p is an odd prime.

Theorem 8. *Let N be a closed topological manifold which is a mod p homology n -sphere, and $T: N \rightarrow N$ be a continuous map of period p without fixed points, where p is an odd prime. Let $f: N \rightarrow M$ be a continuous map of N to a compact orientable topological manifold of dimension m , where $n \geq (p-1)m$. Then the covering dimension of*

$$A(f) = \{y \in N \mid f(y) = f(Ty) = \cdots = f(T^{p-1}y)\}$$

is at least $n - (p-1)m$.

REMARK. Munkholm [5] proved this theorem under a hypotheses that f is a 'nice' map.

Theorem 8 is proved in the similar way to the proof of the main theorem. We shall give outlines of the proof in the following and leave details to the reader.

Let E be a Hausdorff space such that $\tilde{H}_*(E) = 0$, and $T: E \rightarrow E$ be a continuous map of period p without fixed points. Let π denote the cyclic group of order p whose generator is T . Then there exist functorial isomorphisms

$$\begin{aligned} \kappa: H_*(\pi; H_*(X)^p) &\cong H_*(E \times_{\pi} X^p), \\ \kappa: H^*(\pi; H^*(X)^p) &\cong H^*(E \times_{\pi} X^p), \end{aligned}$$

defined for each topological space X such that $H_*(X)$ is free and of finite type (see Theorem 2), and κ preserve the cup products and the cap products (see Theorem 3). In virtue of these results, the elements $P_i(a)$, $P(a_1, \dots, a_p) \in H_*(E \times_{\pi} X^p; \mathbb{Z}_p)$ can be defined for $a, a_1, \dots, a_p \in H_*(X; \mathbb{Z}_p)$, and the elements $P_i(\alpha)$, $P(\alpha_1, \dots, \alpha_p) \in H^*(E \times_{\pi} X^p; \mathbb{Z}_p)$ can be defined for $\alpha, \alpha_1, \dots, \alpha_p \in H^*(X; \mathbb{Z}_p)$. As for these, the theorem similar to Theorem 4 holds. Let $\omega^k \in H^k(E_{\pi}; \mathbb{Z}_p)$ be the canonical generator, and $d^*: H^*(E \times_{\pi} X^p; \mathbb{Z}_p) \rightarrow H^*(E_{\pi} \times X; \mathbb{Z}_p)$ denote the homomorphism induced by the 'diagonal' map $d: E \times X \rightarrow E \times X^p$. Then, for $\alpha \in H^q(X; \mathbb{Z}_p)$ we have

$$d^*P_0(\alpha) = c_q \sum_i (-1)^i (\omega^{(p-1)(q-2i)} \times \mathcal{O}^i(\alpha) - \omega^{(p-1)(q-2i)-1} \times \beta \mathcal{O}^i(\alpha)),$$

where \mathcal{O}^i is the p -th reduced power, β is the Bockstein homomorphism, and

$c_q = (-1)^{q/2}$ or $(-1)^{(q-1)/2}((p-1)/2)!$ according as q is even or odd (see Theorem 5).

Put $N^0 = N$, and define N^{2i} ($i=1, 2, \dots$) inductively to be the join of N^{2i-2} and $S^1 = \{z \in \mathbf{C} \mid |z|=1\}$. We define also N^{2i-1} ($i=1, 2, \dots$) to be a subspace of N^{2i} consisting of all $sy \oplus (1-s)e^{2\pi k \sqrt{-1}/p}$, where $0 \leq s \leq 1$ and $k=0, 1, \dots, p-1$. Let N^∞ be the limit space of the sequence $N \subset N^1 \subset N^2 \subset \dots$. There exists a continuous map $T: N^\infty \rightarrow N^\infty$ of period p without fixed points such that $T(N^i) \subset N^i$ ($i=0, 1, 2, \dots$) and $T|N$ is the given map $T: N \rightarrow N$. In fact, such a map T is defined by

$$T_i(sy \oplus (1-s)e^{2\pi k \sqrt{-1}/p}) = s(T_{i-1}y) \oplus (1-s)e^{2\pi k \sqrt{-1}(t+1/p)}$$

where $T = T|N^{2i}$ and $s, t \in [0, 1]$. It follows that N^∞ has the properties assumed for E , and that $i_*: H_k(N \times X^p; \mathbf{Z}_p) \rightarrow H_k(N^\infty \times X^p; \mathbf{Z}_p)$ is an isomorphism if $k \leq n$ (see Theorem 6).

Let $d_{0*}: H_{n+m}(N_\pi \times M; \mathbf{Z}_p) \rightarrow H_{n+m}(N_\pi \times M^p; \mathbf{Z}_p)$ be the homomorphism induced by the 'diagonal' map, and $\theta_0 \in H^{(p-1)m}(N_\pi \times M^p; \mathbf{Z}_p)$ be the Poincaré dual of $d_{0*}(\lambda)$, where $\lambda \in H_{n+m}(N_\pi \times M; \mathbf{Z}_p)$ is the fundamental class. If $(p-1)m \leq n$, there exists a unique $\theta \in H^{(p-1)m}(N^\infty \times M^p; \mathbf{Z}_p)$ such that $\theta|N_\pi \times M^p = \theta_0$. Similarly to Theorem 7, we have

$$\theta = c_m \left(\sum_j (-1)^j P_{(p-1)(m-2jp)}(U_j) \right) + \delta,$$

where δ is a linear combination of elements of the type $P(\alpha_1, \dots, \alpha_p)$, and $U_j \in H^{2j(p-1)}(M; \mathbf{Z}_p)$ is the "Wu class" defined in terms of \mathcal{P}^j .

Consider now a continuous map $s: N_\pi \rightarrow N_\pi \times M^p$ defined by $s(y) = (y, f(y), f(Ty), \dots, f(T^{p-1}y))$ ($y \in N$), and proceed as in §7. Then we see that $s^*(\theta_0) \neq 0$ and hence $\dim A(f) \geq n - (p-1)m$.

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